Stabilization of Stochastic Quantum Dynamics via Open and Closed Loop Control

QUAINT Workshop, Dijon, April 2013

Claudio Altafini (SISSA)
Kazunori Nishio (Tokyo Institute of Technology)
Francesco Ticozzi (Univ. of Padova)
Introduction

- Stochastic Master Equation (SME) for quantum filtering
- Averaging over noise: Markovian Master Equation (MME)
- Invariance & attractivity of subspaces for MME
  \[ \iff \text{Global Asymptotic Stability (GAS)} \]
- Environment-assisted stabilization:
  - block-structure of the dissipation/measurement operators
  - open-loop control
- Invariance & attractivity of subspaces for the SME
  \[ \iff \text{Global Asymptotic Stability in probability} \]
- Same environment-assisted stabilization properties for MME and SME
- Feedback-assisted stabilization for SME
- Examples
Quantum filtering

- quantum filtering: Stochastic Master Equation (SME) à la Itô:
  - SME

\[
\begin{align*}
  d\rho_t &= \left( \mathcal{F}(H, \rho_t) + \sum_k \mathcal{D}(L_k, \rho_t) + \mathcal{D}(M, \rho_t) \right) dt + \mathcal{G}(M, \rho_t) dW_t
\end{align*}
\]

- \( \rho_t \in \mathcal{D}(\mathcal{H}) \) = set of density operators in \( n \)-dimensional Hilbert space \( \mathcal{H} \)
- continuous weak measurement: output equation

\[
\begin{align*}
  dY_t &= \sqrt{\eta} \frac{1}{2} \text{tr}(\rho_t (M + M^\dag)) dt + dW_t
\end{align*}
\]

- \( M \) = measurement operator \( \in \mathcal{B}(\mathcal{H}) \)
- \( \eta \in [0, 1] \) = efficiency of the measurement
- \( dW_t \) = “innovation process”
- example: homodyne detection
Stochastic Master Equation (SME)

\[ d\rho_t = \left( \mathcal{F}(H, \rho_t) + \sum_k \mathcal{D}(L_k, \rho_t) + \mathcal{D}(M, \rho_t) \right) dt + \mathcal{G}(M, \rho_t) dW_t \]

- Hamiltonian
  \[ \mathcal{F}(H, \rho) := -i[H, \rho] \]

- Lindbladian dissipation
  \[ \mathcal{D}(L_k, \rho) := L_k \rho L_k^\dagger - \frac{1}{2} (L_k^\dagger L_k \rho + \rho L_k^\dagger L_k) \]

- Measurement
  - drift
    \[ \mathcal{D}(M, \rho) := M \rho M^\dagger - \frac{1}{2} (M^\dagger M \rho + \rho M^\dagger M) \]
  - diffusion
    \[ \mathcal{G}(M, \rho) := \sqrt{\eta} (M \rho + \rho M^\dagger - \text{tr}((M + M^\dagger) \rho) \rho) \]
Stochastic Master Equation (SME)

- **infinitesimal generator**

\[ \mathcal{A}[\cdot] = \frac{1}{2} \text{tr} \left( (\mathcal{F}(H, \rho_t) + \sum_k \mathcal{D}(L_k, \rho_t) + \mathcal{D}(M, \rho_t)) \frac{\partial [\cdot]}{\partial \rho} \right. \]

\[ + \left. \frac{\partial [\cdot]}{\partial \rho} (\mathcal{F}(H, \rho_t) + \sum_k \mathcal{D}(L_k, \rho_t) + \mathcal{D}(M, \rho_t)) \right. \]

\[ + \frac{1}{2} G^2(M, \rho_t) \frac{\partial^2 [\cdot]}{\partial \rho^2} + \frac{1}{2} \frac{\partial^2 [\cdot]}{\partial \rho^2} G^2(M, \rho_t) \]

- **solution:**

\[ \rho_t = \mathcal{T}_t^W(\rho_0), \quad \rho_0 \in \mathcal{D}(\mathcal{H}) \]

\[ = \rho_0 + \int_0^t \left( \mathcal{F}(H, \rho_s) + \sum_{k=1}^r \mathcal{D}(L_k, \rho) + \mathcal{D}(M, \rho_s) \right) ds + \int_0^t G(M, \rho_s) dW_s \]

- $\rho_t \exists$ uniquely
- $\rho_t$ adapted to the filtration $\mathcal{E}_t$ associated to $\{W_t, t \in \mathbb{R}^+\}$
- $\rho_t \mathcal{D}(\mathcal{H})$-invariant by construction
Markovian Master Equation (MME)

- averaging the SME over the noise trajectories
- → Markovian Master Equation

\[ \dot{\rho}(t) = \mathcal{L}(\rho(t)) = \mathcal{F}(H, \rho_t) + \sum_k \mathcal{D}(L_k, \rho_t) + \mathcal{D}(M, \rho_t) \]

- Quantum dynamical semigroup \( \mathcal{T}(\rho) \) is a TPCP (Trace-Preserving Completely Positive) map

\[ \rho(t) = \mathcal{T}_t(\rho_0), \quad \rho_0 \in \mathcal{D}(\mathcal{H}) \]

\[ = \exp \int_0^t \left( \mathcal{F}(H, \rho_s) + \sum_{k=1}^r \mathcal{D}(L_k, \rho_s) + \mathcal{D}(M, \rho_s) \right) ds \]

- **Assumption**: Hamiltonian \( H \) can be chosen arbitrarily
State space decomposition


I. State decomposition

\[ \mathcal{H} = \mathcal{H}_S \oplus \mathcal{H}_R \]

- \( \mathcal{H}_S = \) “target” subspace, \( \dim(\mathcal{H}_S) = s \)
- \( \mathcal{H}_R = \) “remainer” subspace, \( \dim(\mathcal{H}_R) = n - s \)
- \( \implies \) block structure for densities in \( \mathcal{D}(\mathcal{H}) \)

\[ \rho = \begin{pmatrix} \rho_S & \rho_P \\ \rho_Q & \rho_R \end{pmatrix} \]

\( \implies \) block structure for operators on \( \mathcal{H} \)

\[ H = \begin{pmatrix} H_S & H_P \\ H_P^\dagger & H_R \end{pmatrix}, \quad L = \begin{pmatrix} L_S & L_P \\ L_Q & L_R \end{pmatrix}, \quad M = \begin{pmatrix} M_S & M_P \\ M_Q & M_R \end{pmatrix} \]
Invariance & attractiveness of $\mathcal{H}_S$ for the MME

$$\mathcal{H} = \mathcal{H}_S \oplus \mathcal{H}_R$$

- density initialized in $\mathcal{H}_S$

$$\mathcal{I}_S(\mathcal{H}) = \left\{ \rho \in \mathcal{D}(\mathcal{H}) \mid \rho = \begin{bmatrix} \rho_S & 0 \\ 0 & 0 \end{bmatrix}, \rho_S \in \mathcal{D}(\mathcal{H}_S) \right\}$$

**Definition:** $\mathcal{H}_S$ is an **invariant** subspace for the system under the TPCP maps $\{T_t(\cdot)\}_{t \geq 0}$ if $\mathcal{I}_S(\mathcal{H})$ is an invariant subset of $\mathcal{D}(\mathcal{H})$, i.e. if

$$\rho \in \mathcal{I}_S(\mathcal{H}) \implies T_t(\rho) \in \mathcal{I}_S(\mathcal{H}) \quad \forall \ t \geq 0$$

**Definition:** $\mathcal{H}_S$ supports an **attractive** subsystem with respect to a family of TPCP maps $\{T_t\}_{t \geq 0}$ if $\forall \rho \in \mathcal{D}(\mathcal{H})$ the following condition is asymptotically obeyed

$$\lim_{t \to \infty} \left\| T_t(\rho) - \begin{bmatrix} \rho_S(t) & 0 \\ 0 & 0 \end{bmatrix} \right\| = 0.$$
Invariance for the MME

**Theorem** \( \mathcal{H}_S \) supports an invariant subspace iff

\[
L_k = \begin{pmatrix}
L_{S,k} & L_{P,k} \\
0 & L_{R,k}
\end{pmatrix}
\quad \forall \ k,
M = \begin{pmatrix}
M_S & M_P \\
0 & M_R
\end{pmatrix}
\]

\[
iH_P - \frac{1}{2} \left( \sum_k L_{S,k}^\dagger L_{P,k} + M_S^\dagger M_P \right) = 0.
\]

**idea of the proof:**

\[
\frac{d}{dt} \rho = \begin{pmatrix}
\mathcal{L}_S(\rho) & \mathcal{L}_P(\rho) \\
\mathcal{L}_Q(\rho) & \mathcal{L}_R(\rho)
\end{pmatrix}
\quad \forall \ \rho_S \in \mathcal{I}_S(\mathcal{H}) \implies \begin{pmatrix}
\mathcal{L}_S(\rho_S) & 0 \\
0 & 0
\end{pmatrix}
\]

**necessary and sufficient conditions on the structure of the blocks of \( H, L_k \) and \( M \)**

- \( M_Q = 0 \) and \( L_{Q,k} = 0 \ \forall k \)
- Hamiltonian \( H_P \) is used to compensate for \( L_{P,k} \neq 0 \) and/or \( M_P \neq 0 \implies \) open-loop “matching”
Attractivity for the MME

**Theorem** Assume $\mathcal{H}_S$ supports an invariant subsystem. Then $\mathcal{I}_S(\mathcal{H})$ can be made attractive iff $\mathcal{I}_R(\mathcal{H})$ is not invariant.

- **idea**: $L_{P,k} \neq 0$ and/or $M_P \neq 0 \implies \mathcal{H}_R$ can never be made invariant by Hamiltonian compensation.

- if $L_{P,k} = L_{Q,k} = 0$ and $M_P = M_Q = 0$ then $\mathcal{I}_S(\mathcal{H})$ and $\mathcal{I}_R(\mathcal{H})$ both invariant (when $H_P = 0$) or non-invariant ($H_P \neq 0$).

- if $L_{P,k} = L_{Q,k} \neq 0$ and/or $M_P = M_Q \neq 0$ then neither $\mathcal{I}_S(\mathcal{H})$ nor $\mathcal{I}_R(\mathcal{H})$ can be invariant.
Attractivity for the MME

**Theorem** Assume $\mathcal{H}_S$ supports an invariant subsystem. Then $\mathcal{I}_S(\mathcal{H})$ can be made attractive iff $\mathcal{I}_R(\mathcal{H})$ is not invariant.

- **idea:** $L_{P,k} \neq 0$ and/or $M_P \neq 0 \Rightarrow \mathcal{H}_R$ can never be made invariant by Hamiltonian compensation
- if $L_{P,k} = L_{Q,k} = 0$ and $M_P = M_Q = 0$ then $\mathcal{I}_S(\mathcal{H})$ and $\mathcal{I}_R(\mathcal{H})$ both invariant (when $H_P = 0$) or non-invariant ($H_P \neq 0$)
- if $L_{P,k} = L_{Q,k} \neq 0$ and/or $M_P = M_Q \neq 0$ then neither $\mathcal{I}_S(\mathcal{H})$ nor $\mathcal{I}_R(\mathcal{H})$ can be invariant

**Summary:** rendering $\mathcal{H}_S$ attractive for the MME is “to the expenses of $\mathcal{H}_R$”, and can be accomplished by means of

1. non-hermitian $L_k$ and/or non-hermitian $M$
   $\Rightarrow$ (block) “ladder-like” operators
2. open-loop control
   - $\mathcal{H}_S$ invariant and attractive $\iff$ Globally Asymptotically Stable (GAS)
   $\Rightarrow$ environment-assisted stabilization
Problem formulation: global asymptotic stability for the SME

\( \mathcal{I}_S(\mathcal{H}) \) globally asymptotically stable in probability

\( \iff \mathcal{I}_S(\mathcal{H}) \) is invariant and attractive in probability a.s.

**Problem** Consider the SME

\[
d\rho_t = \left( \mathcal{F}(H, \rho_t) + \sum_k \mathcal{D}(L_k, \rho_t) + \mathcal{D}(M, \rho_t) \right) dt + \mathcal{G}(M, \rho_t) dW_t
\]

and a target subspace \( \mathcal{H}_S \) such that \( \mathcal{H} = \mathcal{H}_S \oplus \mathcal{H}_R \).

Do there exist

- dissipation operators \( L_k \) and/or measurement operator \( M \)
- Hamiltonian \( H \)

such that the target set \( \mathcal{I}_S(\mathcal{H}) \) is invariant under \( T^W_t(\cdot) \) and attractive in probability a.s.?
Condition for invariance

\[ d\rho = \begin{pmatrix} L_S(\rho) & L_P(\rho) \\ L_Q(\rho) & L_R(\rho) \end{pmatrix} dt + \begin{pmatrix} G_S(M, \rho) & G_P(M, \rho) \\ G_Q(M, \rho) & G_R(M, \rho) \end{pmatrix} dW_t \]

**Approach:** in order for \( \mathcal{I}_S(\mathcal{H}) \) to be invariant for the SME, it has to be invariant for both its diffusion and drift parts

\[ \forall \ \rho_S \in \mathcal{I}_S(\mathcal{H}) \implies \begin{pmatrix} L_S(\rho_S) & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} G_S(M, \rho_S) & 0 \\ 0 & 0 \end{pmatrix} \]

\[ \implies \text{block structure of } L_k \text{ and } M \text{ matrices} \]

**for** \( \rho_S \in \mathcal{I}_S(\mathcal{H}) \)

- diffusion

\[ G(M, \rho_S) = \begin{pmatrix} (M_S - \text{Tr}(M_S\rho_S))\rho_S + \rho_S(M_S^\dagger - \text{Tr}(M_S^\dagger\rho_S)) \\ M_Q\rho_S \\ 0 \end{pmatrix} \begin{pmatrix} \rho_S M_Q^\dagger \\ 0 \end{pmatrix} \]

\[ \implies \mathcal{I}_S(\mathcal{H}) \text{ is invariant to } G(M, \rho_S) \text{ if } M_Q = 0 \]
Condition for invariance

- **drift**

\[
\mathcal{F}(H, \rho_S) + \sum_k \mathcal{D}(L_k, \rho_S) + \mathcal{D}(M, \rho_S) = \\
\begin{bmatrix}
-iH_S\rho_S + i\rho_S H_S + d_1(L_k, M, \rho_S) \\
\quad i\rho_S H_P + \sum_k d_2(L_k, \rho_S) - \frac{1}{2}\rho_S M_S^\dagger M_P \\
\end{bmatrix} \\
\left[ \begin{array}{c}
* \\
\sum_k L_{Q,k} \rho_S L_{Q,k}^\dagger \\
\end{array} \right]
\]

where we define:

\[
d_1(L_k, M, \rho_S) = M_S \rho_S M_S^\dagger - \frac{1}{2}(M_S^\dagger M_S \rho_S + \rho_S M_S^\dagger M_S) + \sum_k \left\{ L_{S,k} \rho_S L_{S,k}^\dagger - \frac{1}{2}(L_{S,k}^\dagger L_{S,k} \rho_S + \rho_S L_{S,k}^\dagger L_{S,k} + L_{Q,k}^\dagger L_{Q,k} \rho_S + \rho_S L_{Q,k}^\dagger L_{Q,k}) \right\},
\]

\[
d_2(L_k, \rho_S) = L_{S,k} \rho_S L_{Q,k}^\dagger - \frac{1}{2}\rho_S (L_{S,k}^\dagger L_{P,k} + L_{Q,k}^\dagger L_{R,k}).
\]

\[\implies\text{same conditions on } H_P, M_P \text{ and } L_{P,k} \text{ as for the MME}\]
Invariance for the SME

- Putting together

**Proposition** \( I_S(\mathcal{H}) \) is invariant for the SME iff

\[
L_k = \begin{pmatrix}
L_{S,k} & L_{P,k} \\
0 & L_{R,k}
\end{pmatrix} \quad \forall k,
\]

\[
M = \begin{pmatrix}
M_S & M_P \\
0 & M_R
\end{pmatrix},
\]

\[
iH_P - \frac{1}{2} \left( \sum_k L_{S,k}^\dagger L_{P,k} + M_S^\dagger M_P \right) = 0,
\]

**Corollary**

\( I_S(\mathcal{H}) \) invariant for the SME \( \iff \) \( I_S(\mathcal{H}) \) invariant for the MME

- More rigorous proof: support theorem
Attractivity for the SME

- attractivity of SME $\iff$ attractivity of MME

**Proposition** Assume $I_S(H)$ is attractive for the MME for some $H$, $L_k$ and $M$. Then with these $H$, $L_k$ and $M$, $I_S(H)$ attractive in probability also for the SME
Attractivity for the SME

- attractivity of SME $\iff$ attractivity of MME

**Proposition** Assume $\mathcal{I}_S(\mathcal{H})$ is attractive for the MME for some $H$, $L_k$ and $M$. Then with these $H$, $L_k$ and $M$, $\mathcal{I}_S(\mathcal{H})$ attractive in probability also for the SME.

**Proof:**
1. stability of $\mathcal{I}_S(\mathcal{H})$ in probability
   - candidate Lyapunov function
     \[
     V(\rho) = \text{tr}(\Pi_R \rho)
     \]
     where $\Pi_R$ = projection on $\mathcal{H}_R$
     \[
     V(\rho) = 0 \text{ for } \rho \in \mathcal{I}_S(\mathcal{H})
     \]
     \[
     V(\rho) > 0 \text{ for } \rho \notin \mathcal{I}_S(\mathcal{H})
     \]
   - \[
   AV(\rho) = -\text{tr}\left(\left(\sum_k L_{P,k}^\dagger L_{P,k} + M_{P}^\dagger M_{P}\right)\rho_R\right) \leq 0 \quad \forall \rho \in \mathcal{D}(\mathcal{H})
   \]
   by ciclicity of the trace
Attractivity for the SME (cont.)

2. attractivity (by contradiction)

- Suppose \( \exists \rho_0 \in D(\mathcal{H}) \setminus I_S(\mathcal{H}) \) for which \( I_S(\mathcal{H}) \) is not attractive

\[
P \left( \lim_{t \to \infty} \varnothing(\mathcal{T}_t^W(\rho_0), I_S(\mathcal{H})) = 0 \right) = 1 - p, \quad p > 0
\]

\[
P \left( \lim_{t \to \infty} V(\mathcal{T}_t^W(\rho_0)) = 0 \right) = 1 - p
\]

\[
\Rightarrow \text{ for } t \to \infty \text{ set } \{ V(\mathcal{T}_t^W(\rho_0)) > 0 \} \text{ has measure } > 0
\]

- in expectation then \( \exists \xi(\rho_0) > 0 \) for which

\[
\limsup_{t \to \infty} E \left[ V(\mathcal{T}_t^W(\rho_0)) \right] = \limsup_{t \to \infty} \int_{\{ V(\mathcal{T}_t^W(\rho_0)) > 0 \}} V(\mathcal{T}_t^W(\rho_0)) dP
\]

\[
\geq \xi(\rho_0) \limsup_{t \to \infty} \int_{\{ V(\mathcal{T}_t^W(\rho_0)) > 0 \}} dP
\]

\[
= \xi(\rho_0) \limsup_{t \to \infty} \{ 1 - P( V(\mathcal{T}_t^W(\rho_0)) = 0 ) \} = \xi(\rho_0)p
\]

\[
\Rightarrow \exists k > 0 \text{ s.t. } \limsup_{t \to \infty} \varnothing( E[\mathcal{T}_t^W(\rho_0)], I_S(\mathcal{H})) \geq \frac{\xi(\rho_0)p}{k} > 0.
\]
Invariance & attractivity of SME

- **Summary:** Invariance and attractivity of MME
  ⇐⇒ invariance and attractivity of the SME
  ⇐⇒ GAS of MME and SME
- both can be accomplished by means of
  1. non-hermitian $L_k$ and/or non-hermitian $M$
     ⇒ (block) “ladder-like” operators
  2. open-loop control

**Theorem** $\mathcal{J}_S(\mathcal{H})$ is GAS in probability for the SME ⇐⇒ $L_k, M$
and $H$ have the structure

\[ L_k = \begin{pmatrix} L_{S,k} & L_{P,k} \\ 0 & L_{R,k} \end{pmatrix}, \quad M = \begin{pmatrix} M_S & M_P \\ 0 & M_R \end{pmatrix}, \]

\[ iH_P - \frac{1}{2} \left( \sum_k L_{S,k}^\dagger L_{P,k} + M_S^\dagger M_P \right) = 0, \]

with at least one $L_{P,k} \neq 0$ and/or $M_P \neq 0$

⇒ environment-assisted stabilization
Block diagonal case

- How about the case

\[ L_k = \begin{pmatrix} L_{S,k} & 0 \\ 0 & L_{R,k} \end{pmatrix} \quad \forall k, \quad M = \begin{pmatrix} M_S & 0 \\ 0 & M_R \end{pmatrix} \]

- Open-loop, time-invariant control: both \( H_S \) and \( H_R \) are invariant \( \implies \) neither is attractive

**Proposition** Given SME with \( L_k \) and \( M \) as above, no time-invariant \( H \) exists rendering \( \mathcal{I}_S(\mathcal{H}) \) GAS in probability

**Proof**: if \( H_P = 0 \) then \( \mathcal{I}_R(\mathcal{H}) \) is invariant; if instead \( H_P \neq 0 \) then \( \mathcal{I}_S(\mathcal{H}) \) cannot be invariant
Feedback-assisted stabilization

**Assumptions:** \( \dim(\mathcal{H}_S) = 1 \) (pure state stabilization)
\[ \dim(\mathcal{H}) \geq 3 \]

**Further split of** \( \mathcal{H}_R \):
\[ \mathcal{H} = \mathcal{H}_S \oplus \underbrace{\mathcal{H}_C \oplus \mathcal{H}_Z}_{\mathcal{H}_R} \]

\[
H = H_c + u(\rho)H_f = \begin{pmatrix}
0 & 0 & 0 \\
0 & H_{c,c} & H_{c,W} \\
0 & H_{c,W}^\dagger & H_{c,z}
\end{pmatrix} + u(\rho) \begin{pmatrix}
0 & H_{f,U} & 0 \\
0 & H_{f,U}^\dagger & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

\[
L_k = \begin{pmatrix}
L_{k,S} & 0 & 0 \\
0 & L_{k,C} & L_{k,W} \\
0 & L_{k,Y} & L_{k,Z}
\end{pmatrix} \quad M = \begin{pmatrix}
M_S & 0 & 0 \\
0 & M_C & M_W \\
0 & M_Y & M_Z
\end{pmatrix}
\]

**Idea:**
- design \( H_{c,W} \) so as to keep "reshuffling" inside \( \mathcal{H}_R = \mathcal{H}_C \oplus \mathcal{H}_Z \)
- use \( H_{f,U} \) to "drain" probability out of \( \mathcal{H}_R \)
Feedback-assisted stabilization

- use "patchy" feedback law similar to
  

**Theorem** Given SME with $L_k$ and $M$ as above and $H = H_c + u(\rho_t)H_f$, where the feedback control law $u(\rho_t)$ s.t. for $\rho_d \in \mathcal{H}_S$

1. If $\text{tr}(\rho_t\rho_d) \geq \gamma$  
   $u(\rho_t) = -\text{tr}(i[H_f, \rho_t]\rho_d)$

2. If $\text{tr}(\rho_t\rho_d) \leq \gamma/2$,  
   $u(\rho_t) = 1$;

3. If $\rho_t \in \mathcal{B} = \{\rho : \gamma/2 < \text{tr}(\rho_t\rho_d) < \gamma\}$, then
   
   - $u(\rho_t) = -\text{tr}(i[H_f, \rho_t]\rho_d)$ if $\rho_t$ last entered $\mathcal{B}$ through the boundary
   
   $\text{tr}(\rho_d\rho) = \gamma$,

   - $u_t = 1$ otherwise.

Then $\exists \gamma > 0$ such that $u(\rho_t)$ renders the SME GAS in probability

- differences with M. Mirrahimi, R. van Handel:
  
  - valid for more general class of Lindbladians
  
  - uses feedback in a "minimal" way (to enable state transitions otherwise impossible)
  
  - uses the environment as much as possible
Example 1: environment only

\[
\rho_d = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad M = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad L_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 2 & 1 \end{bmatrix}
\]

\( \mathcal{H}_S \) is GAS without any open/closed loop Hamiltonian

\[ H_c = 0 \quad H_f = 0 \]

energy population

sample trajectories
Example 2: environment + open loop

\[ \rho_d = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad M = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad L_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \]

\( \mathcal{H}_S \) is rendered GAS by the following open loop Hamiltonian

\[ H_c = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad H_f = 0 \]

energy population  \hspace{1cm} sample trajectories
Example 3: open loop + closed loop

\[
\rho_d = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad L_1 = 0
\]

\[\mathcal{H}_S\] is rendered GAS by the open loop and feedback Hamiltonians

\[
H_c = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad H_f = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\]

energy population

sample trajectories
Conclusion

- Environment (i.e., dissipation) and open-loop control can easily lead to conditions for GAS of a subsystem in both MME and SME.

- Philosophy: make the most use of environment and the least of feedback.

- Crucial ingredients:
  - non-hermitian part ("ladder operator") in the dissipation and/or measurement operator
  - open-loop control design by "matching"

- GAS of MME $\iff$ GAS of SME $\iff$ invariance & attractivity

- When environment and open loop are not enough: feedback can be useful to get GAS